Distribution Theory II

1.1. Structure of Distribution

We continue our discussion on the structure of distributions. Last time we characterized those distributions whose supports are points. Now we consider the general case.

Theorem 1 Let $\Lambda \in \mathcal{D}'(U)$ and $K \subset U$ compact. There exists a multi-index α, $|\alpha| \le n(N+2)$, and some $f \in C(U)$ such that

$$\Lambda \varphi = \int_U f D^\alpha \varphi \ , \quad \forall \varphi \in \mathcal{D}_K .$$

Proof. WLOG assume $K \subset Q$, where $Q = (0,1)^n \subset\subset U$ is the unit cube.

Step 1

$$|\Lambda \varphi| \le C \int |T^{N+1} \varphi(x)| \, d\mathcal{L}^n(x), \quad \forall \varphi \in \mathcal{D}_K, \qquad (1)$$

where $T = D_1 D_2 \cdots D_n$ and N appears in

$$|\Lambda \varphi| \le C \|\varphi\|_N \qquad (2)$$

according to Proposition 1.1 in the previous lecture. First, we claim that

$$\|\varphi\|_N \le \|T^N \varphi\|_0 \ , \quad \forall \varphi \in \mathcal{D}_K . \qquad (3)$$

Clearly, $N=0$ holds. When $N=1$, using $\varphi = 0$ near ∂Q,

$$D_1 \varphi(x) = \int_0^{x_2} D_2 D_1 \varphi(x_1, z_2, x_3, \cdots, x_n) \, dz_2$$

$$= \int_0^{x_2} \int_0^{x_3} D_3 D_2 D_1 \varphi(x_1, z_2, z_3, \cdots x_n) \, dz_2 \, dz_3$$

$$\vdots$$

$$= \int_0^{x_2}\int_0^{x_3}\cdots\int_0^{x_n} D_n D_{n-1}\cdots D_3 D_2 D_1 \varphi(x_1, z_2, \ldots z_n)\, dz_2 \cdots dz_n$$

So

$$\|D_1\varphi\|_{L^\infty} \leq \|T\varphi\|_{L^\infty}.$$

It also holds for D_j, $j \neq 1$, ie

$$\|\varphi\|_1 \leq \|T\varphi\|_{L^\infty}.$$

Assuming $\|\varphi\|_k \leq \|T^k\varphi\|_{L^\infty}$, for α, $|\alpha| = k$,

$$\|D_j D^\alpha \varphi\|_{L^\infty} \leq \|T D^\alpha \varphi\|_{L^\infty} = \|D^\alpha T\varphi\|_{L^\infty} \leq \|T^{k+1}\varphi\|_{L^\infty},$$

So $\|\varphi\|_{k+1} \leq \|T^{k+1}\varphi\|_{L^\infty}$. It follows that (3) holds.

Next, for $\psi \in \mathscr{D}_K$, we have,

$$\psi(x) = \int_{Q(x)} T\psi, \quad \text{where } Q(x) = (0, x_1) \times \cdots \times (0, x_n),$$
$$x = (x_1, \ldots, x_n) \in Q$$

Taking $\psi = T^N\varphi$, and using (2) and (3)

$$|\Lambda\varphi| \leq C\|T^N\varphi\|_{L^\infty}$$
$$\leq C\int_Q |T^{N+1}\varphi|, \quad \forall \varphi \in \mathscr{D}_K. \qquad (4)$$

<u>Step 2</u>.

Let $Y = \{T^{N+1}\varphi : \varphi \in \mathscr{D}_K\} \subseteq L^1(Q)$ a subspace. (4) show that $\varphi \mapsto T^{N+1}\varphi$ is one-to-one, we define

$$\Lambda_1 T^{N+1}\varphi = \Lambda\varphi$$

to obtain a linear functional Λ_1 on Y satisfying

$$|\Lambda_1 T^{N+1}\varphi| \le C \int_Q |T^{N+1}\varphi| .$$

By Hahn-Banach theorem Λ_1 can be extended from Y to $L^1(Q)$ to get $\tilde{\Lambda}_1$ satisfying

$$|\tilde{\Lambda}_1 \varphi| \le C \int_Q |\varphi|, \quad \forall \varphi \in L^1(Q).$$

Using L^1-L^∞ duality, $\exists\, g \in L^\infty(Q)$ s.t.

$$\tilde{\Lambda}_1 \varphi = \int_Q g\varphi, \quad \forall \varphi \in L^1(Q).$$

For $\varphi \in \mathcal{D}_K$, we conclude

$$\Lambda\varphi = \Lambda_1 T^{N+1}\varphi$$
$$= \int_Q g\, T^{N+1}\varphi .$$

<u>Step 3</u> Define

$$f(x) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} g, \quad \text{(extend } g \text{ to } \mathbb{R}^n \text{ by setting it } 0 \text{ outside } Q)$$

then $f \in C(\mathbb{R}^n)$ and $Tf = g$. We have

$$\Lambda\varphi = \int_Q Tf\, T^{N+1}\varphi$$
$$= (-1)^n \int_Q f\, T^{N+2}\varphi, \quad \forall \varphi \in \mathcal{D}_K . \quad \square$$

Corollary 2 Let $\Lambda \in \mathcal{D}'(U)$ be of compact support S_Λ. Then for each V, $S_\Lambda \subset V \subset \bar{V} \subset U$, there exist finitely many continuous fcns f_β in U all supported in V s.t.

$$\Lambda = \sum_\beta D^\beta f_\beta, \quad \text{(a finite sum)}.$$

Pf: Fix some $\psi \in C_c^\infty(U)$, $\psi \equiv 1$ in V. Then, by Theorem 1, for some $f \in C(\mathbb{R}^n)$,

$$\Lambda\varphi = \Lambda(\varphi\psi) \quad \text{(Prop. 1.3 previous lecture)}$$

$$= \int_U f D^\alpha(\varphi\psi)$$

$$= \int_U \sum c_{\beta,\gamma} f D^\beta\psi\, D^\gamma\varphi$$

$$= \int_U \sum_\gamma f_\gamma D^\gamma\varphi, \qquad f_\gamma = \sum_\beta c_{\beta,\gamma} f D^\beta\psi.$$

Corollary 3 Let $\Lambda \in \mathcal{D}'(U)$. There exists continuous fcns $g_\alpha \in C(U)$, one for each index, such that

(a) $\forall$ cpt $K \subset U$, only finitely many g_α are non-zero in K,

(b) $\Lambda = \sum_\alpha D^\alpha g_\alpha$.

Proof. Let $\{B_j\}$ be all balls contained in U so $U = \bigcup B_j$. Let $\{\psi_j\}$ be a P.U. subordinate to $\{B_j\}$ which is locally finite.

Then $\psi_j \Lambda$ is compactly supported in B_j, so Corollary 1 implies that

$$\psi_j \Lambda = \sum_\alpha D^\alpha f_{j,\alpha} \quad \text{(a finite summation).}$$

Let

$$g_\alpha = \sum_{j=1}^{\infty} f_{j,\alpha} \in C(\mathbb{R}^n).$$

(The sum is finite for each $x \in U$.) Then

$$\begin{aligned}
\Lambda\varphi &= \sum_j \Lambda(\psi_j \varphi) \\
&= \sum_j \sum_\alpha (-1)^\alpha \int f_{j,\alpha} D^\alpha \varphi \\
&= \sum_\alpha (-1)^\alpha \int g_\alpha D^\alpha \varphi \\
&= \Big(\sum_\alpha D^\alpha g_\alpha\Big) \varphi .
\end{aligned}$$

□

<u>Remark</u> Λ is <u>of order</u> N if

$$|\Lambda\varphi| \le C \|\varphi\|_N, \quad \forall \varphi \in \mathcal{D}(U).$$

When Λ is of order N, there are only finitely many non-zero g_α so that

$$\Lambda = \sum_\alpha D^\alpha g_\alpha$$

is a finite sum.

1.2 Convolution of a distribution and a test function.

Recall that for $f, g \in L^1(\mathbb{R}^n)$,

$$(f*g)(x) = \int f(x-y)\, g(y)\, d\mathcal{L}^n(y)$$
$$= \int f(y)\, g(x-y)\, d\mathcal{L}^n(y).$$

We learned that

- $x \longmapsto (f*g)(x)$ is $\mathcal{L}^n$-measurable,
- $\|f*g\|_{L^1} \leq \|f\|_{L^1} \|g\|_{L^1}$.

Now, let u, v, etc be $\mathcal{D}' = \mathcal{D}'(\mathbb{R}^n)$ and $\varphi, \psi \in \mathcal{D} = \mathcal{D}(\mathbb{R}^n)$.

Introduce

$$(\tau_x \varphi)(y) = \varphi(y-x),$$
$$\check{\varphi}(y) = \varphi(-y),$$
$$(\tau_x u)(\varphi) = u(\tau_{-x}\varphi) \quad (\text{so } \tau_x u \in \mathcal{D}')$$
$$(u*\varphi)(x) = u(\tau_x \check{\varphi}) \quad (\text{so } u*\varphi \text{ is a fcn in } x)$$

The last definition is inspired by, when u is a function,

$$(u*\varphi)(x) = \int u(y)\, \varphi(x-y)\, d\mathcal{L}^n(y)$$
$$= \int u(y) (\tau_x \check{\varphi})(y)\, d\mathcal{L}^n(y).$$

<u>Proposition 4</u>

$$\tau_x(u*\varphi) = (\tau_x u)*\varphi = u*(\tau_x\varphi)$$

Proof. Note that

$$y \mapsto (u*\varphi)(y) = u(\tau_y \check{\varphi})$$

is a function. therefore,

$$\tau_x(u*\varphi)(y) = (u*\varphi)(y-x)$$
$$= u(\tau_{y-x}\check{\varphi}),$$

$$(\tau_{y-x}\check{\varphi})(z) = \check{\varphi}(z+x-y)$$
$$= \varphi(-z-x+y).$$

Next,

$$(\tau_x u)*\varphi(y) = (\tau_x u)(\tau_y\check{\varphi})$$
$$= u(\tau_{-x}\tau_y\check{\varphi}),$$

$$(\tau_{-x}\tau_y\check{\varphi})(z) = \tau_y\check{\varphi}(z+x)$$
$$= \check{\varphi}(z+x-y)$$
$$= \varphi(-z-x+y).$$

Last,

$$(u*\tau_x\varphi)(y) = u(\tau_y \check{\tau_x\varphi})$$

$$\tau_y\check{\tau_x\varphi}(z) = \check{\tau_x\varphi}(z-y)$$
$$= \tau_x\varphi(y-z)$$
$$= \varphi(y-z-x).$$

So, they are equal. $\square$

Proposition 5 $u*\varphi \in C^\infty(\mathbb{R}^n)$, and

$$D^\alpha(u*\varphi) = (D^\alpha u)*\varphi = u*(D^\alpha\varphi)$$

for any index α.

Proof. It suffices to show

$$D_j(u*\varphi) = (D_j u)*\varphi = u*(D_j\varphi), \quad j=1,\cdots n.$$

First,

$$D_j(u*\varphi)(x) = \lim_{t\to 0} \frac{(u*\varphi)(x+te_j)-(u*\varphi)(x)}{t}.$$

We've

$$\frac{1}{t}\left((u*\varphi)(x+te_j)-(u*\varphi)(x)\right) = \frac{1}{t}\left(u(\tau_{x+te_j}\check{\varphi}) - u(\tau_x\check{\varphi})\right)$$

$$= u\left(\frac{\tau_{x+te_j}\check{\varphi}-\tau_x\check{\varphi}}{t}\right). \qquad (5)$$

Consider

$$\frac{(\tau_{x+te_j}\check{\varphi})(z)-(\tau_x\check{\varphi})(z)}{t} = \frac{\varphi(x-z+te_j)-\varphi(x-z)}{t}$$

$$\to D_j\varphi(x-z) \quad \text{in } \mathcal{D}$$

(here x is fixed). Therefore, by the continuity of u and (5),

$$\frac{1}{t}\left((u*\varphi)(x+te_j)-(u*\varphi)(x)\right) \to u(D_j\varphi(x-z)) = u(\tau_x \check{D_j\varphi})$$

$$= u*D_j\varphi(x).$$

This is the equality between the first and the third term.

On the other hand,

$$(D_j u) * \varphi (x) = D_j u(\tau_x \check{\varphi})$$
$$= -u(D_j \tau_x \check{\varphi}).$$

$$(D_j \tau_x \check{\varphi})(z) = \lim_{t \to 0} \frac{\tau_x \check{\varphi}(z + te_j) - \tau_x \check{\varphi}(z)}{t}$$
$$= \lim_{t \to 0} \frac{\varphi(x - z - te_j) - \varphi(x - z)}{t}$$
$$= -D_j \varphi(x - z).$$

$$\therefore (D_j u) * \varphi(x) = -u(D_j \tau_x \check{\varphi})$$
$$= u(D_j \varphi(x - z))$$
$$= u(\tau_x \check{D_j \varphi})$$
$$= u * D_j \varphi(x), \text{ done. } \square$$

<u>Proposition 6</u> $(u * \varphi)(\psi) = u(\check{\varphi} * \psi)$, $u \in \mathcal{D}'$, $\varphi, \psi \in \mathcal{D}$.

Proof. Let $\{P_n\}$ be a sequence of partitions whose length $\to 0$.

then
$$(\check{\varphi} * \psi)(y) = \int \varphi(x - y)\psi(x)\,dx$$
$$= \lim_{\|P_n\| \to 0} \sum \varphi(x_j - y)\psi(x_j)\Delta x_j \quad (\text{in } \mathcal{D})$$

By continuity,
$$u(\check{\varphi} * \psi) = \lim_{\|P_n\| \to 0} \sum u(\varphi(x_j - y))\psi(x_j)\Delta x_j$$

$$= \lim_{\|P_n\| \to 0} \sum u(\tau_{x_j} \check{\varphi}) \psi(x_j) \Delta x_j$$

$$= \int u(\tau_x \check{\varphi}) \psi(x) \, d\mathcal{L}^n(x)$$

$$= (u * \varphi)(\psi). \qquad \square$$

Let $h \in C_c^\infty(\mathbb{R}^n)$, $0 \le h \le 1$, satisfy $\int h = 1$.

Then $\{h_j\}$, $\quad h_j(x) = j^n h(jx)$, $\int h_j = 1$,

is called <u>approximate identity</u>.

<u>Proposition 7</u> $\forall \varphi \in \mathcal{D}$, $u \in \mathcal{D}'$

(a) $\varphi * h_j \to \varphi$ in $\mathcal{D}$,

(b) $u * h_j \to u$ in $\mathcal{D}'$ (ie, $(u * h_j)(\varphi) \to u(\varphi)$ $\forall \varphi \in \mathcal{D}$.)

Proof. (a) is well-known. For (b), we use Prop. 6,

$$(u * h_j)(\varphi) = u(\check{h}_j * \varphi).$$

By (a), $\check{h}_j * \varphi \to \varphi$ in $\mathcal{D}$, so

$$u(\check{h}_j * \varphi) \to u(\varphi) \text{ in } \mathcal{D}', \text{ done.} \qquad \square$$

<u>Remark</u> (b) shows that every distribution can be approximated by smooth fcns in $*$-weak sense.

1.3 Fourier Transform on the Schwartz space

Let
$$dm_n = \frac{1}{(2\pi)^{n/2}} d\mathcal{L}^n$$

and redefine convolution

$$(f * g)(x) = \int f(x-y)\, g(y)\, dm_n(y).$$

Set
$$e_t(x) = e^{it\cdot x}.$$

For complex-valued $f \in L^1(\mathbb{R}^n)$, define its Fourier transform to be
$$\hat{f}(x) = \int f(y)\, e^{-ix\cdot y}\, dm_n(y)$$
$$= \int f\, e_{-x}\, dm$$
$$= (f * e_x)(0).$$

A basic result is

<u>Theorem 8.</u> $f \in L^1(\mathbb{R}^n)$ implies $\hat{f} \in C_0(\mathbb{R}^n)$.

I leave the proof as a exercise.

<u>Proposition 9</u> $f \in L^1(\mathbb{R}^n)$

(a) $(\tau_x f)^\wedge = e_{-x}\hat{f}$,

(b) $(e_x f)^\wedge = \tau_x \hat{f}$,

Proof (a) $f \in L^1(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, clearly $\tau_x f \in L^1$ so $(\tau_x f)\hat{}$ makes sense. Now

$$\begin{aligned}
(\tau_x f)\hat{}(y) &= \int (\tau_x f)(z)\, e^{-iz\cdot y}\, dm(z) \\
&= \int f(z-x)\, e^{-iz\cdot y}\, dm(z) \\
&= \int f(z-x)\, e^{-i(z-x)\cdot y}\, dm(z)\, e^{-ix\cdot y} \\
&= \hat{f}(y)\, e_{-x}(y). \ \#
\end{aligned}$$

(b) $|e_x| = 1$, so $e_x f \in L^1$ if $f \in L^1$.

$$\begin{aligned}
(e_x f)\hat{}(y) &= \int e_x(z) f(z)\, e^{-iz\cdot y}\, dm(z) \\
&= \int f(z)\, e^{-iz\cdot(y-x)}\, dm(z) \\
&= \hat{f}(y-x) \\
&= \tau_x \hat{f}\,(y). \ \#
\end{aligned}$$

<u>Proposition 10</u> $f, g \in L^1$,

$$(f*g)\hat{} = \hat{f}\,\hat{g}$$

Proof. $f, g \in L^1 \Rightarrow f*g \in L^1$. Now

$$\begin{aligned}
(f*g)\hat{}(x) &= \int (f*g)(y)\, e^{-ix\cdot y}\, dm(y) \\
&= \int\int f(z) g(y-z)\, dm(z)\, e^{-ix\cdot y}\, dm(y) \qquad \text{(Fubini)} \\
&= \int f(z) \int g(y-z)\, e^{-ix\cdot y}\, dm(y)\, dm(z)
\end{aligned}$$

$$= \int f(z) \int g(y-z)\, e^{-ix\cdot(y-z)}\, dm(y)\, e^{-ix\cdot z}\, dm(z)$$

$$= \hat{f}(x)\, \hat{g}(x). \quad \square$$

As $\mathcal{D} \subset L^1$, Fourier transform is well-defined on the space of test functions. A drawback, however, is that $\hat{\mathcal{D}} \not\subset \mathcal{D}$. It is easy to see that the support of a function could lose compactness after Fourier transform. We need to enlarge $\mathcal{D}$ to $\mathcal{S}$, the Schwartz space so that $\hat{\mathcal{S}} \subset \mathcal{S}$. (Later we'll see $\hat{\mathcal{S}} = \mathcal{S}$.)

A smooth fcn $\varphi \in \mathcal{S}$ if $\forall m, N\ \exists\ C(m,N)$ s.t.

$$|D^\alpha \varphi(x)| \leq \frac{C(m,N)}{(1+|x|^2)^{m/2}}, \quad \forall \alpha,\ |\alpha| \leq N, \quad \forall x \in \mathbb{R}^n.$$

then the norms

$$\{\|\varphi\|_{m,N}\}_{m,N\geq 0}$$

makes $\mathcal{S}$ into a locally convex topological space. $\{\varphi_n\} \subset \mathcal{S}$ is convergent to $\varphi \in \mathcal{S}$ iff, for all (m,N),

$$\|\varphi_n - \varphi\|_{m,N} \to 0 \quad \text{as } n \to \infty.$$

In particular, on each compact K,

$$\|\varphi_n - \varphi\|_{N,K} \to 0 \quad \text{as } n \to \infty$$

<u>Proposition 11</u> $\mathcal{D} \subset \mathcal{S}$ <u>and</u> $\varphi_n \to \varphi$ <u>in</u> $\mathcal{D}$ <u>implies</u> $\varphi_n \to \varphi$ in $\mathcal{S}$.

Proof. $\mathcal{D} \subset \mathcal{S}$ is clear. On the other hand, if $\varphi_n \to \varphi$ in $\mathcal{D}$, $\exists$ cpt K s.t. φ_n, φ all vanish outside K, so $\varphi_n \to \varphi$ in $\mathcal{S}$ too. $\square$

Proposition 13 $\mathcal{S}$ is a Fréchet space.

Pf: Exercise.

Proposition 14 Let p be a polynomial, $\psi \in \mathcal{S}$. Then

$$\begin{aligned} \varphi &\longmapsto \psi \varphi \\ &\longmapsto p\varphi \\ &\longmapsto D^\alpha \varphi \end{aligned}$$

are continuous linear maps from $\mathcal{S}$ to $\mathcal{S}$.

Pf: Exercise.

We introduce the notation

$$D_\alpha = \left(\frac{1}{i} D_1\right)^{\alpha_1} \cdots \left(\frac{1}{i} D_n\right)^{\alpha_n}, \quad \alpha = (\alpha_1, \ldots \alpha_n)$$

and, for a polynomial $P(\xi) = \sum_\gamma c_\gamma \xi^\gamma$,

$$P(D) = \sum_\gamma c_\gamma D^\gamma.$$

We have

$$P(D) e_t = P(t) e_t$$

Proposition 15 (a) $\widehat{(P(D)f)} = P\hat{f}$

(b) $\widehat{(Pf)} = P(-D)\hat{f}$

Proof.

$$\widehat{(P(D)f)}(t) = \int \sum c_\gamma D^\gamma f(x) e^{-ix\cdot t} dm(x)$$

$$= \sum (-1)^{|\gamma|} c_\gamma \int f(x) D^\gamma e^{-ix\cdot t} dm(x)$$

$$= \sum (-1)^{|\gamma|} c_\gamma (-i)^{|\gamma|} \int f(x) t^\gamma e^{-ix\cdot t} dm(x)$$

$$= P(t) \hat{f}(t).$$

$$P(-D)\hat{f}(t) = \sum c_\gamma (-D)^\gamma \int f(x) e^{-ix\cdot t} dm(x)$$

$$= \sum c_\gamma \int f(x) x^\gamma e^{-ix\cdot t} dm(x)$$

$$= \widehat{Pf}(t). \quad \square$$

<u>Corollary</u> 16. $\hat{\mathcal{S}} \subset \mathcal{S}$.

Proof. For $\varphi \in \mathcal{S}$, need to show $\forall l, N$,

$$\sup_t (1+|t|^2)^l D^\alpha \hat{\varphi}(t) < \infty, \quad |\alpha| \le N.$$

Indeed, using Proposition 15

$$(1+|t|^2)^l D^\alpha \hat{\varphi}(t) = (1+|t|^2)^l \widehat{(-x)^\alpha \varphi(x)}(t)$$

$$= \widehat{P(D)(-x)^\alpha \varphi(x)}(t)$$

where $P(x) = (1+|x|^2)^l$. Since

$$P(D)(-x)^\alpha \varphi(x) \in \mathcal{S} \subset L^1$$

$$\widehat{P(D)(-x)^\alpha \varphi(x)} \in C_0(\mathbb{R}^n)$$

so

$(1+|t|^2)^l D^\alpha \hat{\varphi}(t)$ is bounded in $\mathbb{R}^n$. $\square$